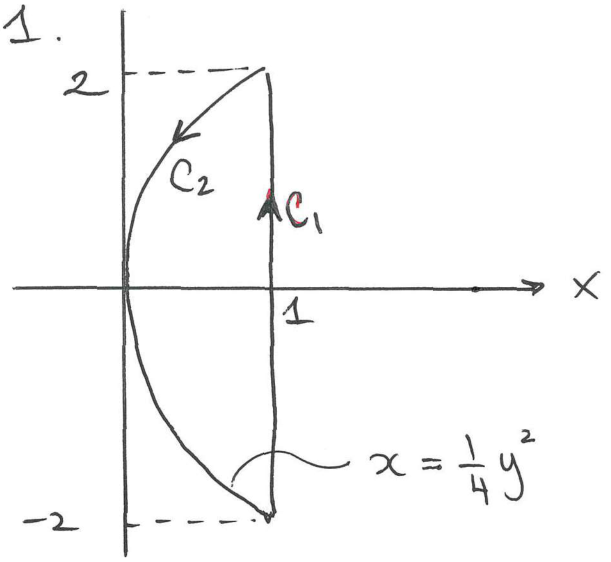
Problem Set 4.

#1.



a)

$\underline{F} = (x^2y, xy^2)$.

Parameterize:

$C_1 : \underline{r}_1(t) = (1, -2+4t)$
$t \in [0,1]$.

$C_2 : \underline{r}_2(t) = (\frac{1}{4}t^2, t)$
$t \in [2,-2]$.

b) $I_1 = \int_{C_1} \underline{F} \cdot \underline{r}_1'(t) \cdot dt = \int_0^1 \left(1\cdot(-2+4t),\ 1\cdot(-2+4t)^2\right) \cdot (0,4) \cdot dt$

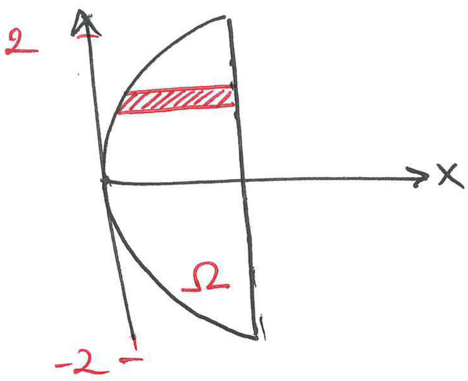
$= \int_0^1 4(-2+4t)^2 \cdot dt = \frac{16}{3}$

$I_2 = \int_{C_2} \underbrace{\left(\left(\frac{1}{4}t^2\right)^2 t,\ \left(\frac{1}{4}t^2\right)\cdot t^2\right)}_{\underline{F}(\underline{r}_2(t))} \cdot \underbrace{\left(\frac{1}{2}t,\ 1\right)}_{\underline{r}_2'(t)} \cdot dt$

$= \int_2^{-2} \left\{\frac{t^4}{4} + \frac{t^6}{32}\right\} \cdot dt = -\frac{152}{35}$.

$\therefore I_1 + I_2 = \frac{104}{105}$

(c). $I = \iint_{\Omega} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \cdot dA$



$= \iint_{\Omega} (y^2 - x^2) \cdot dx \cdot dy$

note Ω given by $-2 \le y \le 2$

$\frac{y^2}{4} \le x \le 1$

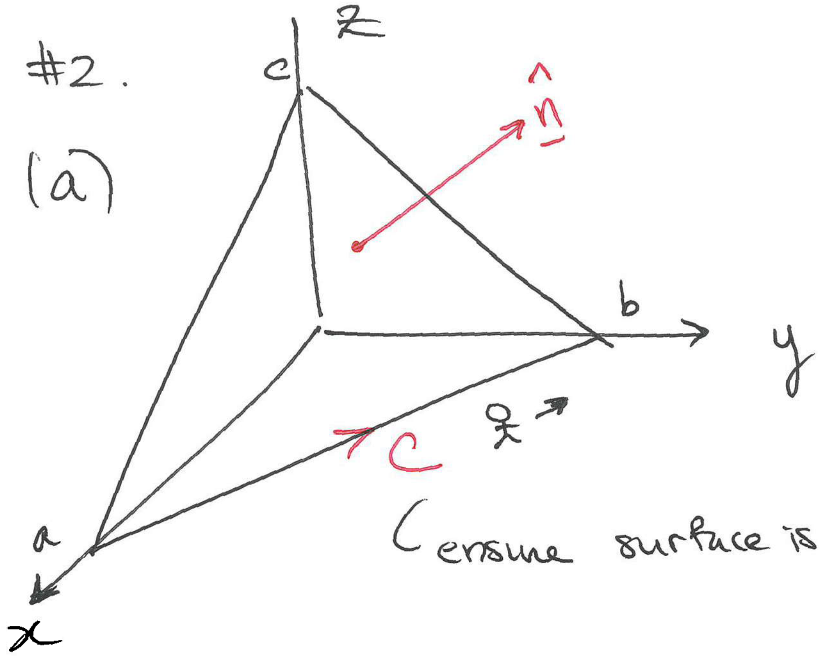
$\therefore \; I = \int_{-2}^{2} \int_{y^2/4}^{1} (y^2 - x^2) \cdot dx \cdot dy = \frac{104}{105}$

Check

$\therefore \iint_{\Omega} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C \underline{F} \circ d\underline{s}$

#2.

(a)



C ensure surface is to the left

(b) * This question is slightly mistaken as I intended to note, via PS #2, Q4, that $a=3$, $b=3$, $c=6$.

From $\underline{F} = (xz^2, -2xy, yz)$

$$\nabla \times \underline{F} = (z, 2xz, -2y)$$

Thus if we want $\oint_C \underline{F} \circ d\underline{r}$, we can instead calculate $\iint_S (\nabla \times \underline{F})\, d\underline{S}$

You did exactly this in PS 2 # 3, where

$$\iint_S (z, 2xz, -2y) \circ \hat{n}\, dS = 36$$

#3. (a) Show $\int_a^b \phi \psi_x \, dx = [\phi\psi]_a^b - \int_a^b \psi \phi_x \cdot dx$

(This is just integration by parts...)

note $\frac{d}{dx}(\phi\psi) = \phi\psi_x + \phi_x \psi$

$\therefore$ LHS $= \int_a^b \left\{ \frac{d}{dx}(\phi\psi) - \phi_x \psi \right\} \cdot dx = [\phi\psi]_a^b - \int_a^b \psi \phi_x \cdot dx$

$= \text{RHS}.$ $\square$

(b) Div. Thm: $\iiint_\Omega \nabla \cdot \underline{G} \cdot dV = \iint_{\partial\Omega} \underline{G} \cdot \underline{\hat{n}} \, dS$

Let $\underline{G} = \phi \underline{F}$

$$\text{LHS} = \iiint_\Omega \nabla \cdot (\phi \underline{F}) \, dV = \iiint_\Omega \{ \nabla\phi \cdot \underline{F} + \phi (\nabla \cdot \underline{F}) \} \, dV$$

$$= \iiint_\Omega \phi (\nabla \cdot \underline{F}) \, dV + \iiint_\Omega (\nabla\phi \cdot \underline{F}) \, dV$$

But LHS $= \iint_{\partial\Omega} (\phi \underline{F}) \cdot \underline{\hat{n}} \, dS$ by Div. Thm

Thus

$$\iiint_\Omega \phi (\nabla \cdot \underline{F}) \, dV = -\iiint_\Omega (\nabla\phi \cdot \underline{F}) \, dV + \iint_{\partial\Omega} (\phi \underline{F}) \cdot \underline{\hat{n}} \, dS$$

This is multi-dimensional version of integ. by parts.

(c)

You just proved: $\int_{\Omega} \phi(\nabla\cdot \underline{F})\,dV = \int_{\partial\Omega} (\phi \underline{F})\cdot \hat{\underline{n}}\,dS - \int_{\Omega} (\nabla\phi\cdot \underline{F})\,dV$

You want to show : $\int_{\Omega} u\nabla^2 v\,dV = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\,dS - \int_{\Omega} (\nabla u\cdot\nabla v)\,dV$

So choose $\phi = u$, $\underline{F} = \nabla v$

notice $(\phi\,\underline{F})\cdot\hat{\underline{n}} = u(\nabla v\cdot\hat{\underline{n}})$

$= u\,\frac{\partial v}{\partial n}$.

The rest follows immediately.

PS 4: Supplementary.

#4. Show area of curve C is

$$A = \frac{1}{2}\oint x \cdot dy - y \cdot dx.$$

By Green's Thm:

$$\oint_C x \cdot dy - y \cdot dx = \oint_C (-y, x) \cdot (dy, dx)$$

$$= \oint_C \underline{F} \cdot d\underline{r}$$

where $\underline{F} = (-y, x)$,

Then $I = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$

$$= \iint_D (2) \cdot dx \cdot dy$$

$$\therefore \iint_D dx \cdot dy = \frac{1}{2}\oint_C F \cdot dr$$

as desired.

For ellipse: $x = a\cos\theta$, $y = b\sin\theta$

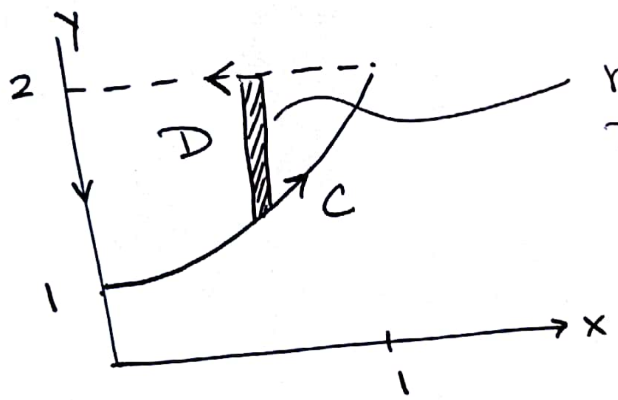
$$\therefore \text{area} = \frac{1}{2}\int_{\theta=0}^{2\pi} (-b\sin\theta, a\cos\theta) \cdot r'(\theta)\, d\theta$$

$$\therefore \text{ area} = \frac{1}{2}\int_0^{2\pi} (-b\sin\theta, a\cos\theta)\cdot(-a\sin\theta, b\cos\theta)\, d\theta$$

$$= \frac{ab}{2}(2\pi) = \underline{\underline{ab\pi}}.$$

#5. Calculate $\int_C \underline{F}\cdot d\underline{r}$ where $\underline{F} = (x\cos y, x^2\sin y)$

C is boundary of region $\{(x,y) \mid 1+x^2 \le y \le 2, x \geq 0\}$



note
$D = \{(x,y) \mid 0 \le x \le 1,\ 1+x^2 \le y \le 2\}$

By Green's: $\oint_C \underline{F}\cdot d\underline{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

so we need

$$I = \iint_D (2x\sin y + x\sin y)\, dx\, dy$$

$$= \iint_D (3x\sin y)\, dx\, dy.$$

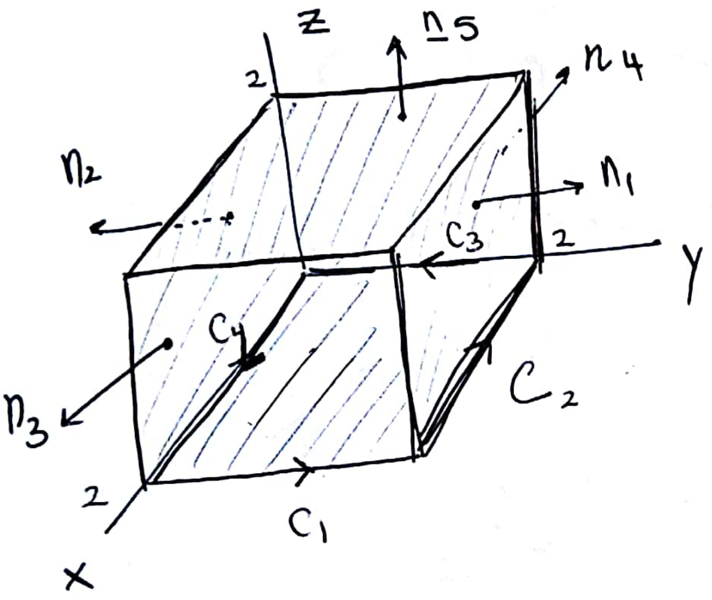
$$= \int_{x=0}^{1}\int_{y=1+x^2}^{2} \{3x\sin y\}\, dx\, dy = \underline{\underline{-\frac{3}{2}\cos 2 + \frac{3}{2}\{\sin 2 - \sin 1\}}}$$

#6. Verify $\iint_S \nabla \times \underline{F} \cdot d\underline{S} = \oint_C \underline{F} \cdot d\underline{r}$

where $\underline{F} = (y - z + 2,\ yz + 4,\ -xz)$

S = surface of cube $[0,2]^3$

with open base on $z = 0$



So $S = S_1 \cup \dots \cup S_5$

where each normal is:

$\hat{\underline{n}}_1 = (0,1,0)$ & S_1 has $y=2$, $x,z \in [0,2]$

$\hat{\underline{n}}_2 = (0,-1,0)$ & S_2 has $y=0$, $x,z \in [0,2]$

$\hat{\underline{n}}_3 = (1,0,0)$ & S_3 has $x=2$, $y,z \in [0,2]$

$\hat{\underline{n}}_4 = (-1,0,0)$ & S_4 has $x=0$, $y,z \in [0,2]$

$\hat{\underline{n}}_5 = (0,0,1)$ & S_5 has $z=2$, $x,y \in [0,2]$

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Similarly:

$C = C_1 \cup C_2 \cup C_3 \cup C_4$

where $C_1 : \{ x=2, z=0, \; 0 \le y \le 2 \}$

$C_2 : \{ x \text{ from } 2 \text{ to } 0, \; y=2, \; z=0 \}$

$C_3 : \{ y \text{ from } 2 \text{ to } 0, \; x=0, \; z=0 \}$

$C_4 : \{ 0 \le x \le 2, \; y=0, \; z=0 \}$

$$\int_{C_1} \underline{F} \cdot d\underline{r} = \int_{y=0}^{2} (y+2, 4, 0) \cdot (0, dy, 0)$$

$$= \int_0^2 4 \cdot dy = 8.$$

$$\int_{C_2} \underline{F} \cdot d\underline{r} = \int_{t=0}^{2} (4, 4, 0) \cdot (-1, 0, 0) \, dt = -8$$

use $\underline{r}(t) = (2-t, 0, 0)$

$$\int_{C_3} \underline{F} \cdot d\underline{r} = \int_{t=0}^{2} (y+2, 4, 0) \cdot (0, -1, 0) \, dt = -8$$

use $\underline{r}(t) = (0, 2-t, 0)$

$$\int_{C_4} \underline{F} \cdot d\underline{r} = \int_{x=0}^{2} (2, 4, 0) \cdot (dx, 0, 0) = 4$$

$$\therefore \oint_C \underline{F} \circ d\underline{r} = -4$$

Check other values on surface integral:

$$\iint_S \nabla \times \underline{F} \cdot d\underline{S} = \left(\iint_{S_1} + \cdots + \iint_{S_5} \right) \nabla \times \underline{F} \, d\underline{S}$$

Then $\nabla \times \underline{F} = (-y, z-1, -1)$

On S_1: $(-y, z-1, -1) \cdot (0,1,0) = z-1$ with $y=2$

S_2: $(-y, z-1, -1) \cdot (0,-1,0) = -(z-1)$ with $y=0$

S_3: $(-y, z-1, -1) \cdot (1,0,0) = -y$ with $x=2$

S_4: $(-y, z-1, -1) \cdot (-1,0,0) = y$. with $x=0$

S_5: $(-y, z-1, -1) \cdot (0,0,1) = -1$ with $z=2$

now

$$\iint_{S_1} (z-1) \, dS + \iint_{S_2} (-(z-1)) \, dS = 0$$

on account of the y-independence.

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If you don't believe it...

$$\underbrace{\int_{x=0}^{2}\int_{z=0}^{2}(z-1)\cdot dz\cdot dx}_{S_1 \text{ has } y=2} - \underbrace{\int_{x=0}^{2}\int_{z=0}^{2}(z-1)\cdot dz\cdot dx}_{S_2 \text{ has } y=0} = 0.$$

Similarly,

$$-\iint_{\substack{S_3\\(x=2)}} y\cdot dy\, dz + \iint_{\substack{S_4\\x=0}} y\cdot dy\cdot dz = 0.$$

Thus, we need only calculate

$$\iint_{\substack{S_5\\z=2}} (-1)\, dS = \int_{x=0}^{2}\int_{y=0}^{2}(-1)\, dS = \underline{\underline{-4}}$$

$\therefore$ Stokes Thm verified