PROBLEM SET #3.

#1. (i) $\nabla \cdot \underline{F} = 5xy$, $\nabla \times \underline{F} = (xz, -yz, y^2 - x^2)$

(ii) $\nabla \cdot \underline{F} = 0$, $\nabla \times \underline{F} = (x, -y, -x\cos y)$

#2. Prove $\nabla \cdot (\phi \underline{F}) = (\nabla \phi) \cdot \underline{F} + \phi (\nabla \cdot \underline{F})$

$$
\begin{aligned}
\text{LHS} &= \partial_x (\phi F_1) + \partial_y (\phi F_2) + \partial_z (\phi F_3) \\
&= \{\phi_x F_1 + \phi_y F_2 + \phi_z F_3\} + \phi \{F_{1x} + F_{2y} + F_{3z}\} \\
&= \nabla \phi \cdot \underline{F} + \phi (\nabla \cdot \underline{F}) \\
&= \text{RHS}
\end{aligned}
$$

<u>NB</u>: Index notation:

$$\text{LHS} = \partial_i (\phi F_i) = (\partial_i \phi) F_i + \phi (\partial_i F_i) = (\nabla \phi) \cdot \underline{F} + \phi (\nabla \cdot \underline{F})$$

$\square$

Prove $\nabla \times (\phi \underline{F}) = (\nabla \phi) \times \underline{F} + \phi (\nabla \times \underline{F})$

$$
\text{LHS} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \phi F_1 & \phi F_2 & \phi F_3 \end{vmatrix} = \big(\partial_y(\phi F_3) - \partial_z(\phi F_2),\ \partial_z(\phi F_1) - \partial_x(\phi F_3),\ \partial_x(\phi F_2) - \partial_y(\phi F_1)\big).
$$

Examine first component for example.

$$\{\phi_y F_3 - \phi_z F_2\} + \{F_{3y} - F_{2z}\}\phi,$$

which you can check is the same as

$$(\partial \phi \times F)_1 + \phi (\nabla \times \underline{F})_1$$

Alternatively, index notation:

$$LHS = (\nabla \times (\phi F))_i = \varepsilon_{ijk}\, \partial_j (\phi F)_k$$

$$= \varepsilon_{ijk} (\partial_j \phi) F_k + \varepsilon_{ijk}\, \phi\, \partial_j F_k$$

$$= (\nabla\phi \times \underline{F})_i + \phi (\nabla \times \underline{E})_i$$

$$= RHS$$

(See how easy it is?) $\square$

(c) If $\nabla^2 \phi = 0$, show $\nabla \cdot (\nabla \phi) = 0$ and $\nabla \times (\nabla \phi) = 0$

The fact $\nabla \cdot (\nabla \phi) = 0$ follows from def'n of ∇^2. So,

$$\nabla \cdot (\nabla \phi) = (\partial x, \partial y, \partial z) \cdot (\phi_x, \phi_y + \phi_z)$$

$$= \phi_{xx} + \phi_{yy} + \phi_{zz}$$

$$= \nabla^2 \phi = 0.$$

Prove $\nabla \times (\nabla \phi) = 0$:

$$LHS = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ \phi_x & \phi_y & \phi_z \end{vmatrix} = \begin{pmatrix} \phi_{yz} - \phi_{zy} \\ \phi_{zx} - \phi_{xz} \\ \phi_{xy} - \phi_{yx} \end{pmatrix} = 0$$

due to equality of mixed partials.

#3.

(a). Define $\underline{r}(\theta, z) = (a\cos\theta,\ a\sin\theta,\ z)$

Then check $(\underline{r}_\theta \times \underline{r}_z) = (a\cos\theta,\ a\sin\theta,\ 0)$

Thus $$\underline{\hat{n}} = \frac{\underline{r}_\theta \times \underline{r}_z}{|\underline{r}_\theta \times \underline{r}_z|} = \frac{(x,\ y,\ 0)}{\sqrt{x^2+y^2}}$$

and $dS = |\underline{r}_\theta \times \underline{r}_z| \cdot d\theta\, dz = a \cdot d\theta\, dz$.

(b). Verify $\displaystyle\int_V \nabla \cdot \underline{F}\, dV = \int_S \underline{F} \cdot \underline{\hat{n}}\, dS$.

$$\text{LHS} = \int_{z=0}^{3} \int_{\rho=0}^{2} \int_{\theta=0}^{2\pi} \{4 - 4\rho\sin\theta + 2z\}\, \rho \cdot d\theta \cdot d\rho\, dz = \underline{\underline{84\pi}}$$

$$\text{RHS} = \left(\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} \right) \underline{F} \cdot \underline{\hat{n}}\, dS$$

$\hat{n}_{\text{top}} = (0,0,1) \Rightarrow \underline{F} \cdot \underline{\hat{n}} = z^2 \big|_{z=3} = 9$

$\hat{n}_{\text{bottom}} = (0,0,-1) \Rightarrow \underline{F} \cdot \underline{\hat{n}} = -z^2 \big|_{z=0} = 0.$

Thus $\displaystyle\int_{\text{top}} \underline{F} \cdot \underline{\hat{n}}\, dS = 9 \cdot \int_{\text{top}} dS = 9 \cdot (\pi a^2) = 36\pi$

where $a = 2$.

This leaves

$$\int_{\text{sides}} \underline{F} \cdot \underline{\hat{n}} \, dS = \int_{\theta=0}^{2\pi} \int_{z=0}^{3} \left(\frac{4a^2 \cos^2\theta - a^3 \cdot 2 \cdot \sin^3\theta}{a} \right) a \cdot dz \cdot d\theta$$

$$= 3 \cdot 16 \cdot \pi$$

Altogether, $\int_S \underline{F} \cdot \underline{\hat{n}} \, dS = 84\pi$ $\square$.

#4. Prove $\nabla \cdot (\underline{F} \times \underline{G}) = \underline{G} \cdot (\nabla \times \underline{F}) - \underline{F} \cdot (\nabla \times \underline{G})$

Index notation saves time!

$$LHS = \partial_i (\underline{F} \times \underline{G})_i = \partial_i (\varepsilon_{ijk} F_j G_k)$$

$$= G_k [\varepsilon_{ijk} \partial_i F_j] + F_j [\varepsilon_{ijk} \partial_i G_k]$$

now $[\varepsilon_{ijk} \partial_i F_j] = \varepsilon_{kij} \partial_i F_j$ (cyclic)

$$= (\nabla \times \underline{F})_k$$

$[\varepsilon_{ijk} \partial_i G_k] = - \varepsilon_{jik} \partial_i G_k$ (acyclic)

$$= -(\nabla \times \underline{G})_j$$

$$\therefore LHS = (\nabla \times \underline{F})_k G_k - (\nabla \times \underline{G})_j F_j$$

$$= RHS$$

$\square$

#5. By Q4:

$$\nabla \cdot (\underline{c} \times \underline{x}) = c \cdot \underbrace{(\nabla \times \underline{x})}_{=0} - \underline{x} \cdot \underbrace{(\nabla \times \underline{c})}_{=0 \text{ since } \underline{c} \text{ is constant}}$$

$$= 0.$$

Next for $\nabla \times (\underline{c} \times \underline{x})$ do manually:

$$\underline{c} \times \underline{x} = \begin{vmatrix} i & j & k \\ c_1 & c_2 & c_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \begin{pmatrix} c_2 x_3 - x_2 c_3 \\ x_1 c_3 - c_1 x_3 \\ c_1 x_2 - x_1 c_2 \end{pmatrix}$$

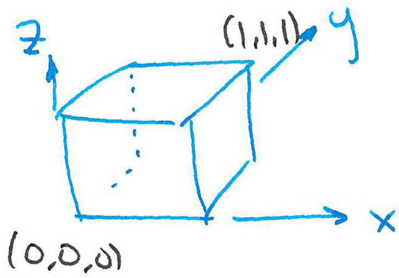
Then

$$\nabla \times (\underline{c} \times \underline{x}) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ c_2 x_3 - x_2 c_3 & x_1 c_3 - c_1 x_3 & c_1 x_2 - x_1 c_2 \end{vmatrix}$$

$$= 2 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 2 \underline{c}.$$

$\square$

6.



Check: $\nabla \cdot \underline{F} = 3xy(\cdot x + y) + x(1 - 6y) + 2z$

$$\int_V \nabla \cdot \underline{F} \cdot dV = 1.$$

also $$\int_S \underline{F} \cdot \hat{n}\, dS = \left(\int_{\text{face } y=0} + \int_{\text{face } y=1} + \int_{\text{face } x=0} + \int_{\text{face } x=1} + \int_{\text{face } z=0} + \int_{\text{face } z=1} \right) \underline{F} \cdot d\underline{S}$$

$$= 0 + 0 + 0 + 0 + 1 + 0 \quad = 1$$

as required.