Given a Fourier Series on $[-\pi, \pi]$, we approximate $f(x)$ using partial sums,

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \{a_n \cos(nx) + b_n \sin(nx)\}.$$

and define

$$S(x) \equiv \lim_{N\to\infty} S_N(x).$$

Q1. Does $N \to \infty$ make sense?

Q2. Does $S(x) = f(x)$?

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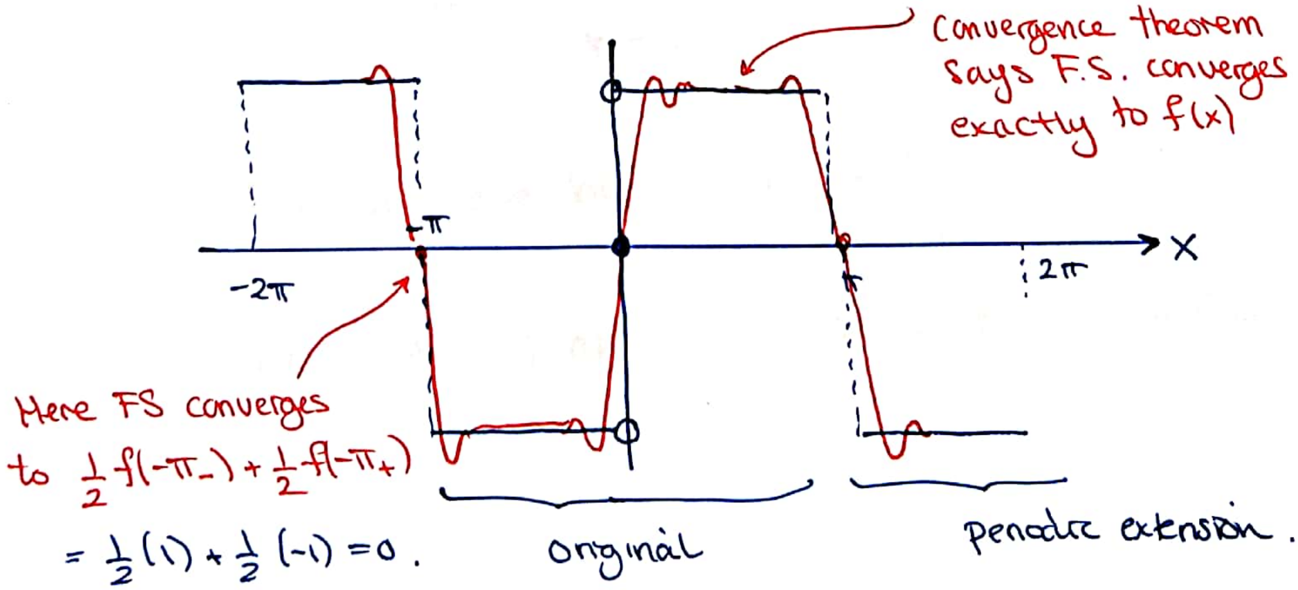
* We call the individual sines/cosines "modes".

N modes = N "cosines/sines".

Example 12.1 (Square wave)

$$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi \\ 0 & \text{if } x = 0, \pi \end{cases}$$



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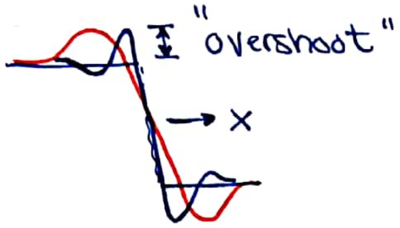
We will the finite overshoot that exists as $N \to \infty$ the Gibbs Phenomenon.



Based on the experiment, we conclude that (it appears) $S_N(x) \to f(x)$ as $N \to \infty$ pointwise (for fixed x). But it does not do so uniformly.

Def'n 12.2 (One sided limit)

The one-sided limit from the right is,

$$f(c_+) = \lim_{\substack{h \to 0 \\ h > 0}} f(c+h).$$

Similarly from the left,

$$f(c_-) = \lim_{\substack{h \to 0 \\ h > 0}} f(c-h)$$

E.g. The sq. wave has $f(0_+) = 1$
$f(0_-) = -1$

and $\sin\left(\frac{1}{x}\right)$ as neither $\pm$ limits as $x \to 0_\pm$.

Here $a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{odd}} \underbrace{\cos(nx)}_{\text{even}} \cdot dx = 0. \quad \forall\, n \geq 0.$

$$b_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \underbrace{f(x)\sin(nx)}_{\text{even}}\, dx \qquad n \geq 1$$

$$= \frac{2}{\pi} \cdot \int_{0}^{\pi} (1)\sin(nx) \cdot dx$$

$$= -\frac{2}{n\pi} \cos(nx) \Big|_0^{\pi} = -\frac{2}{n\pi} \Big\{ \underbrace{\cos(n\pi)}_{(-1)^n} - \underbrace{\cos(0)}_{1} \Big\}$$

$$= \frac{2}{n\pi} \left\{ 1 - (-1)^n \right\}.$$

$$\therefore\ f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[1 - (-1)^n \right] \cdot \sin(nx).$$

$$= \sum_{k=0}^{\infty} \frac{2 \cdot 2}{(2k+1)\pi} \sin\left[(2k+1)x\right]. \qquad n = 2k+1$$

Def'n 12.4 (Piecewise cts.)

f is piecewise cts on (a,b) if (a,b) can be subdivided into finite # of intervals where f is cts, and moreover, the right- and left-sided limits exist everywhere.

Thm 12.5 (Fourier convergence theorem)

Let f be 2π-periodic on $(-\pi,\pi)$ with f and f' piecewise cts. Then the F.S. of f at x converges to the average of the left- and right limits.

$$\Rightarrow \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\} = \frac{1}{2} \cdot \left\{ f(x_-) + f(x_+) \right\}$$