We would like to represent general functions as an infinite series of sines + cosines:

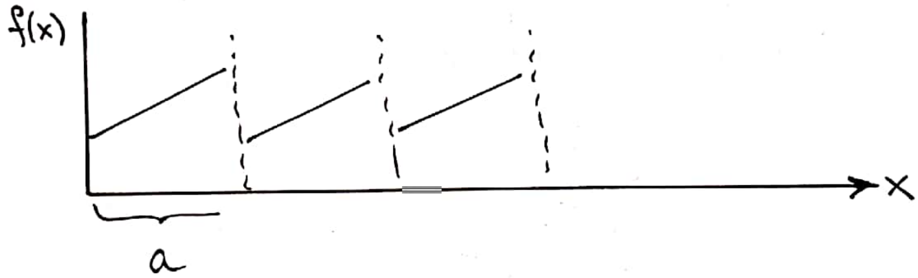
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\}.$$

Two questions:

(i) How to calculate a_n, b_n?

(ii) To what extent is "$\sim$" an equality $=$?

Def'n 11.1: A function is periodic with period a if $f(x+a) = f(x)$.

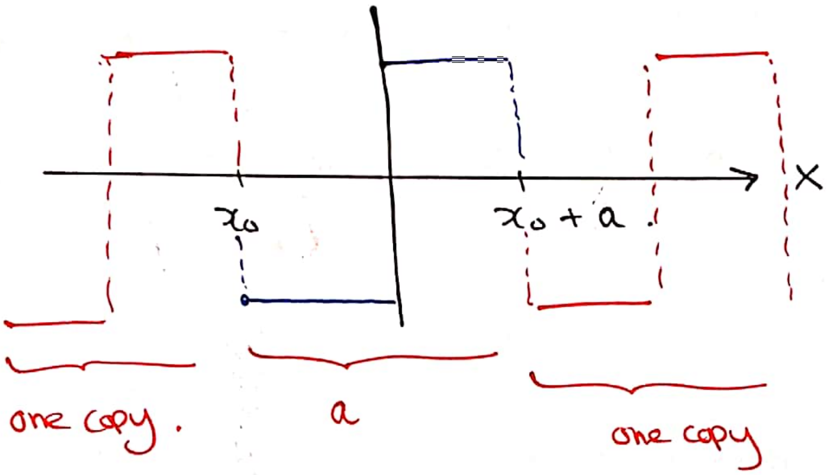


The smallest such 'a' is the fundamental period.

Example:
- $\sin(kx)$ is $\frac{2\pi}{k}$-periodic
- x, x^2, e^x are not periodic.
- is $e^{\cos x}$ periodic? (No).

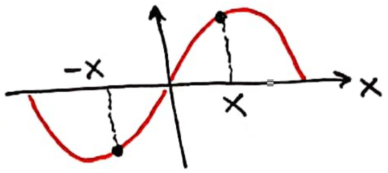
Def'n 11.3: (Periodic extension).

Consider $f(x)$ on $(x_0, x_0 + a)$. This can be extended to $\mathbb{R}$ via a a-periodic extension.



Def'n 11.4 (odd function).

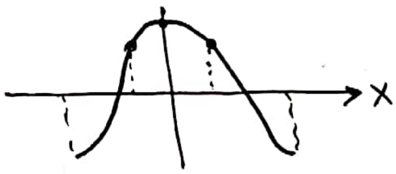
$f(x)$ is odd if $f(-x) = -f(x)$.



$\sin(-x) = -\sin(x)$.

Def'n 11.5 (even function)

$f(x)$ is even if $f(-x) = f(x)$.



$\cos(-x) = \cos(x)$

Examples:
- $x^2, x^4, 1, \ldots$ are even
- $x^3, x^5, x \ldots$ are odd
- e^x is neither even or odd.

Lemma 11.6 :

1. $f(0) = 0$ if f is odd.
2. If f is even, then $\int_{-a}^{a} f(x) \cdot dx = 2 \cdot \int_{0}^{a} f(x)\, dx$
3. If f is odd, then $\int_{-a}^{a} f(x)\, dx = 0$.

Lemma 11.8 If $m, n \in \mathbb{Z}$, with $n, m \in \mathbb{Z} \setminus \{0\}$, then:

$$\int_{-\pi}^{\pi} \sin(nx) \cos(mx) \cdot dx = 0. \qquad (1)$$

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) \cdot dx = \pi \cdot \delta_{mn} \qquad (2)$$

$$\int_{-\pi}^{\pi} \sin(nx) \sin(mx)\, dx = \pi\, \delta_{mn} \qquad (3)$$

$$\delta_{mn} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m. \end{cases}$$

* note that the case with $n=0, m=0$ done separately.

Pf : See notes. Let's prove (2).

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) \cdot dx = \int_{-\pi}^{\pi} \frac{1}{2} \cdot \left\{ \cos(n-m)x + \cos(n+m)x \right\} dx$$

[cf (11.9a)-(11.9c)].

$$= \frac{1}{2} \cdot \left[\frac{\sin(n-m)x}{n-m} + \frac{\sin(n+m)x}{n+m} \right]_{-\pi}^{\pi} \quad \text{if } n \neq m.$$

$$= 0.$$

If $n = m \neq 0$, then $I = \int_{-\pi}^{\pi} \frac{1}{2} \cdot \{\cos(0) + \cos(2nx)\} \, dx$

$$= \pi$$

If $n = m = 0$, then $I = \int_{-\pi}^{\pi} \frac{1}{2} \cdot \{\cos(0) + \cos(0)\} \, dx$

$$= 2\pi$$

So $\int_{-\pi}^{\pi} \cos(nx)\cos(mx) \cdot dx = \begin{cases} 0 & \text{if } n \neq m. \\ \pi & \text{if } n = m \neq 0. \\ 2\pi & \text{if } n = m = 0. \end{cases}$

There is neat way to think.

Define an inner product between $f(x), g(x)$:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \cdot dx$$

So in essence,

- $\langle \sin(nx), \cos(mx) \rangle = 0 \Rightarrow$ sin is $\perp$ (orthogonal) to cos.
- $\langle \sin(nx), \sin(mx) \rangle = 0$ if $n \neq m \Rightarrow$ sin $\perp$ to other sin
- $\langle \sin(nx), \sin(nx) \rangle = \pi \Rightarrow \sin(nx)$ is $/\!/$ to $\sin(nx)$.

Idea: $f(x) \sim \underbrace{\frac{a_0}{2}}_{V_0} + (\underbrace{a_1 \cos(x)}_{a_1 V_1} + \underbrace{b_1 \sin x}_{b_1 V_2}) + (a_2 \cos(2x) + \cdots)$ ...

$\langle f, v_1 \rangle \sim \underbrace{\langle \frac{a_0}{2} V_0, V_1 \rangle}_{0} + a_1 \langle V_1, V_1 \rangle + \underbrace{\langle V_1, b_1 V_2 \rangle}_{=0} + \cdots$

Remember if $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$ form an orthogonal basis. Seek coefs a_i s.t.

$$\underline{v} = a_1 \underline{u}_1 + \cdots + a_n \underline{u}_n$$

Dot product: $\underline{v} \cdot \underline{u}_m = a_1 \underline{u}_1 \cdot \underline{u}_m + \cdots + a_m \underline{u}_m \cdot \underline{u}_m + \cdots + a_n \underline{u}_n \cdot \underline{u}_m .$

But orthogonality $\Rightarrow \underline{u}_n \cdot \underline{u}_m = 0$ if $n \neq m$.

$$\therefore \quad \underline{v} \cdot \underline{u}_m = a_m (\underline{u}_m \cdot \underline{u}_m)$$

$$\therefore \quad a_m = \frac{\underline{v} \cdot \underline{u}_m}{\|\underline{u}_m\|^2}$$

The same idea is applied to derive the Fourier coefs. We seek a_i and b_i s.t.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\}$$

Let's multiply by $\cos(mx)$ and $\int_{-\pi}^{\pi}$:

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \cdot \cos(mx) \cdot dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(mx) \cdot dx$$

$$+ \int_{-\pi}^{\pi} \left[a_1 \cos(x) \cos(mx) + a_2 \cos(2x) \cos(mx) + \cdots \right] \cdot dx$$

$$+ \int_{-\pi}^{\pi} \left[b_1 \sin(x) \cos(mx) + b_2 \sin(2x) \cdot \cos(mx) + \cdots \right] \cdot dx$$

where $m = 0, 1, \ldots, n$.

If $m=0$, we have

$$\int_{-\pi}^{\pi} f(x)\cdot dx = \frac{a_0}{2}\cdot\int_{-\pi}^{\pi}(1)\cdot dx + [0+\cdots+0].$$

$$\Rightarrow \qquad a_0 = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(x)\cdot dx.$$

If $m \neq 0$, we have by orthogonality.

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos(mx)\cdot dx = 0 + \left[0+\cdots+a_m\int_{-\pi}^{\pi}\cos^2 mx\, dx + \cdots + 0\right] + [0+\cdots+0].$$

But $\int_{-\pi}^{\pi}\cos^2 mx\cdot dx = \pi$

$$\therefore \quad a_m = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(x)\cos(mx)\cdot dx \qquad m \geq 1$$

Similar procedure yields $b_m = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(x)\sin(mx)\, dx$

$m \geq 1$

<u>THM 11.9</u> (FOURIER COEFFICIENTS) Let f be a 2π periodic function defined on $[-\pi,\pi]$.

Then we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos(nx) + b_n\sin(nx)\}$$

where,

$$a_n = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(x)\cos(nx)\cdot dx \qquad n \geq 0.$$

$$b_n = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx \qquad n \geq 1$$

* We write $\sim$ because we're not sure about the convergence properties.

In certain cases, our Fourier Series reduces.

Lemma 11.11 (Fourier Sine or Cosine Series).

If $f(x)$ is even on $[-\pi, \pi]$, then

$$\begin{cases} f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx). \\ a_n = \frac{2}{\pi} \cdot \int_0^{\pi} f(x) \cos(nx) \cdot dx. & n \geq 0. \\ b_n = 0. & n \geq 1. \end{cases}$$

If $f(x)$ is odd on $[-\pi, \pi]$,

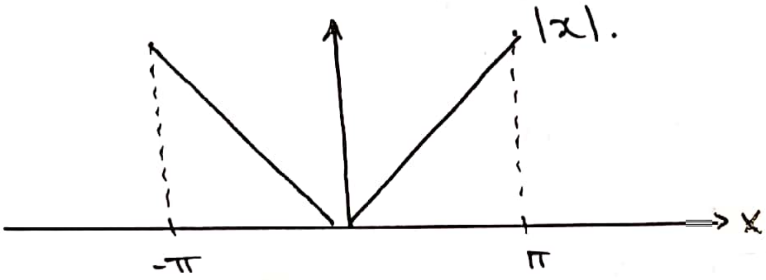
$$\begin{cases} f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx). \\ a_n = 0. & n \geq 0. \\ b_n = \frac{2}{\pi} \cdot \int_0^{\pi} f(x) \sin(nx) & n \geq 1. \end{cases}$$

Pf (cf. notes)

e.g. f even then $\int_{-\pi}^{\pi} \underbrace{f(x)\sin(nx)}_{\text{odd function.}} \cdot dx = 0.$

Example 11.12. Find the Fourier Series of

$f(x) = |x|$ on $x \in [-\pi, \pi]$.



Write $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos(nx) + b_n \sin(nx) \}$.

<u>Compute</u>: $a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cos(nx) \cdot dx$.

$= \frac{2}{\pi} \cdot \int_0^{\pi} x \cdot \cos(nx) \cdot dx$.

* Let $u = x$, $dv = \cos(nx)\,dx$
I.B.P.

$= \frac{2}{\pi} \cdot \left\{ x \left(+ \frac{\sin(nx)}{n} \right) \Big|_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n} \cdot dx \right\}$ $(n \neq 0)$

$= \frac{2}{\pi} \left\{ 0 + \frac{1}{n^2} \cos(nx) \Big|_0^{\pi} \right\}$ since $\sin(n\pi) = 0$

$= + \frac{2}{n^2 \pi} \left\{ \cos(n\pi) - 1 \right\}$.

$= + \frac{2}{n^2 \pi} \left\{ (-1)^n - 1 \right\}$

$a_n = + \frac{2}{n^2 \pi} \cdot \begin{cases} 0 & \text{if } n \text{ even} \\ +2 & \text{if } n \text{ odd.} \end{cases}$ $(n \neq 0)$

If $n=0 \Rightarrow a_0 = \frac{2}{\pi} \cdot \int_0^{\pi} x \cdot dx = \frac{2}{\pi} \cdot \frac{1}{2} \pi^2 = \pi$

What about b_n ? $\quad b_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} |x| \cdot \sin(nx) \cdot dx$

$= 0$ since $|x| \sin(nx)$ is odd.

$$\therefore \quad |x| \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{+2}{n^2 \pi} \right] \left[(-1)^n - 1 \right] \cdot \cos(nx).$$

$$= \frac{\pi}{2} + \sum_{\substack{n=1 \\ n \text{ odd.}}}^{\infty} \left[\frac{-4}{n^2 \pi} \right] \cos(nx)$$

Let $n = 2k+1 \quad k = 0, 1, 2, \ldots$

$$\therefore \quad |x| \sim \frac{\pi}{2} + \sum_{k=0}^{\infty} \left[\frac{-4}{(2k+1)^2 \pi} \right] \cos\left[(2k+1) x \right].$$

Example: Find FS for $f(x) = \begin{cases} 0 & -5 < x < 0. \\ 3 & 0 < x < 5 \end{cases}$

Note the $f(x)$ is on $[-L, L]$, $L \neq \pi$.

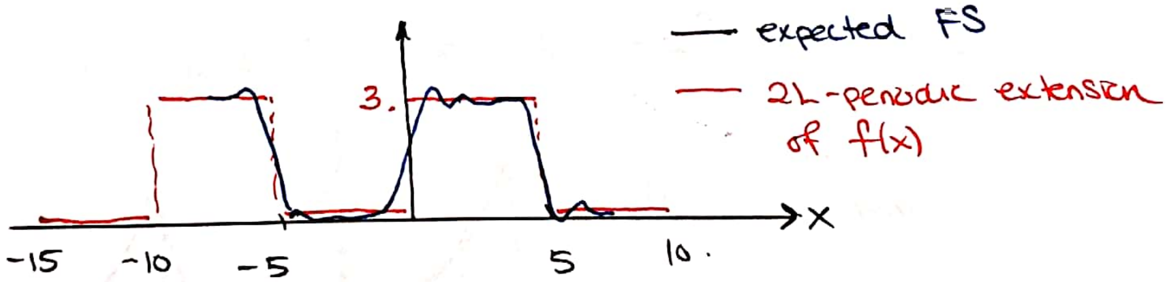
In fact the general formulae for $[-L, L]$ is:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

where $a_n = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \cdot dx \qquad n \geq 0.$

* $b_n = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \cdot dx \qquad n \geq 1$

(See next chapter. for details.)



* note if $L = \pi \Rightarrow$ reduces to previous formulae.

We have,

$$a_n = \frac{1}{5} \cdot \int_{-5}^{5} f(x) \cos\left(\frac{n\pi x}{5}\right) dx$$

$$= \frac{1}{5} \cdot \left(\int_{-5}^{0} (0) \cos\left(\frac{n\pi x}{5}\right) \cdot dx + \int_{0}^{5} 3 \cdot \cos\left(\frac{n\pi x}{5}\right) \cdot dx \right)$$

$$= \frac{3}{5} \cdot \left(\frac{5}{n\pi}\right) \sin\left(\frac{n\pi x}{5}\right) \Big|_{0}^{5} = (0 - 0) \quad \text{since } \sin(n\pi) = 0.$$

$(n \neq 0)$

$$a_0 = \frac{1}{5} \cdot \int_0^5 3\cos(0) \cdot dx = 3.$$

$$b_n = \frac{1}{5} \cdot \int_0^5 3\sin\left(\frac{n\pi x}{5}\right) \cdot dx = \frac{3}{5}\left(-\frac{5}{n\pi}\right)\cos\left(\frac{n\pi x}{5}\right)\Bigg|_0^5$$

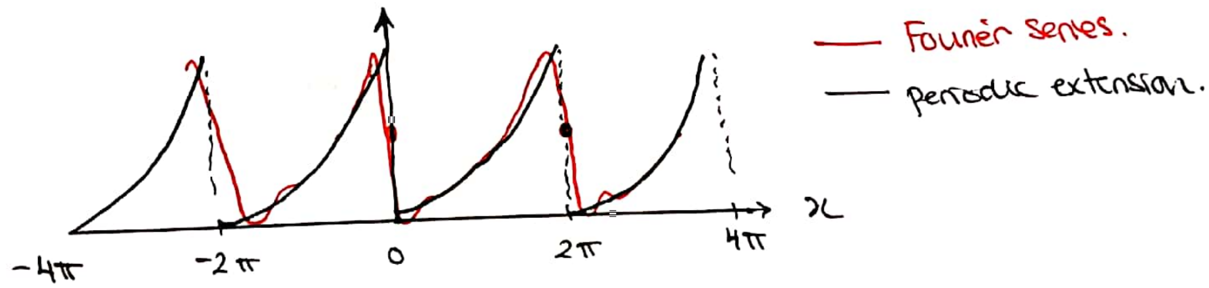
$$= -\frac{3}{n\pi} \cdot \left\{\cos(n\pi) - \cos(0)\right\}$$

$$= -\frac{3}{n\pi}\left\{(-1)^n - 1\right\}.$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{6}{(2k+1)\pi} & \text{if } n = 2k+1 \text{ (odd)} \\ & k = 0, 1, 2, \dots \end{cases}$$

$$\therefore f(x) \sim \frac{3}{2} + \sum_{k=0}^{\infty} \left[\frac{6}{(2k+1)\pi}\right] \sin\left[\frac{(2k+1)\pi x}{5}\right].$$

Example: Calculate FS for $f(x) = x^2$ on $x \in [0, 2\pi]$.



The Fourier series will still be, for a $2L$ periodic function,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right\}$$

$$a_n = \frac{1}{L} \cdot \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) \cdot dx \qquad n \geq 0$$

$$b_n = \frac{1}{L} \cdot \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 1$$

(This is Remark. 12.8)

Here $2L = 2\pi \Rightarrow L = \pi$.

$$a_n = \frac{1}{\pi} \cdot \int_0^{2\pi} f(x) \cos\left(\frac{n\pi x}{\pi}\right) \cdot dx$$

$$= \frac{1}{\pi} \cdot \int_0^{2\pi} x^2 \cos(nx) \cdot dx$$

$$= \frac{1}{\pi} \cdot \left\{ x^2 \left(\cancel{\frac{\sin(nx)}{n}}\right) \Big|_0^{2\pi} \right.$$

$u = x^2$
$dv = \cos(nx) \cdot dx$

0.

$$\left. - \int_0^{2\pi} (2x) \left(\frac{\sin nx}{n}\right) \cdot dx \right\}$$ $(n \neq 0)$

$$= -\frac{2}{\pi} \frac{1}{n} \cdot \left\{ x \left(-\frac{\cos nx}{n}\right) \Big|_0^{2\pi} + \int_0^{2\pi} (1) \cancel{\left(-\frac{\cos nx}{n}\right)} \cdot dx \right\}$$

0

$$= \frac{2}{n^2 \pi} \left\{ (2\pi) \cos(2n\pi) - 0 \right\}$$

$$a_n = \frac{4}{n^2} \quad \text{if} \quad n \neq 0.$$

(To finish Tues.)